PMM U.S.S.R.,Vol.46,pp. 553-555 0021-8928/83/4 0553 $7.50/0
Copyright Pergamon Press Ltd.1983.Printed in U.K.
UDC 62-50

# ON A PROBLEM OF ESCAPE ALONG A PRESCRIBED CURVE*

A. AZAMOV

A differential game is considered, in which a pursuer can move over the whole plane with unit velocity. while the escaper can move along a prescribed curve with a bounded velocity greater than unity. An escape strategy is constructed, ensuring a positive constant lower bound for the distance between the players.

**1. Statement of the problem.** A regular curve $\Gamma$ of smoothness class $C^2$ is prescribed on a plane, along which an escaper moves with a velocity not exceeding $\sigma, \sigma > 1$. The pursuer, whose velocity does not exceed unity in absolute value, moves over the whole plane and tries to overtake the escaper. A further refinement of the problem statement (informativeness and class of admissible strategies of the escaper, the concept of escape possibility, etc.) can be traced from the proof of the fundamental theorem in Sect.4. The paper's purpose is to construct an escape strategy permitting evasion from capture, as well as to derive an estimate for the distance between the pursuer and escaper. It is close in spirit to /1,2/. An escape problem in the case when there are an arbitrary number of pursuers, while the escaper moves in a neighborhood of an arbitrarily prescribed stright line, was solved in /1/. The existence of optimal strategies for both players was proved in /2/ in the case when the curve $\Gamma$ is a circle. If in the game at hand the curve $\Gamma$ is replaced by a graph, then once again a nontrivial game arises.

**2. Preliminary constructions.** We introduce the following assumptions.

**Assumption A.** The curvature of curve $\Gamma$ is bounded (in modulus) by a number $1/\rho$, $\rho > 0$. If $\rho \geqslant 1$, then it is obvious that the curvature is bounded also by the number $1$, so that we can set $\rho = 1$. Therefore, without loss of generality we can take $\rho \leqslant 1$ or replace $\rho$ by $\min\{1, \rho\}$.

**Assumption B.** Curve $\Gamma$ either is closed or is of infinite length in both directions from each of its points.

Obviously, if curve $\Gamma$ has a finite length, even if in one direction only, then for specific dispositions of the players at the start of the game the pursuer overtakes the escaper in finite time. See Sect.5 for the motivation of Assumption A. This Assumption enables us to localize the problem. Let $Q_0$ be an arbitrary point on curve $\Gamma$ and $\Pi$ be a rectangle centered at point $Q_0$, two sides of which are parallel to the tangent to curve $\Gamma$ at point $Q_0$ and are of length $k\rho$, while the other pair of sides are half the size. Here and subsequently, $k$ is a positive number depending only on $\rho$ and $\sigma$ and specially selected, while $k \leqslant 1/\sqrt{5}$.

**Lemma 1.** In the neighborhood $\Pi$ of each point $Q_0 \in \Gamma$ the curve $\Gamma$ can be represented, in an appropriate coordinate system as the graph of a function $y = f(x)$, $|x| \leqslant k\rho$, where $f \in C^2$, $f(0) = f'(0) = 0$. The inequalities

$$|f'(x)| \leqslant |x|\cdot(\rho^2 - x^2)^{-1/2} \leqslant k(1 - k^2)^{-1/2} \tag{2.1}$$

$$|x| \leqslant s(x) = \int_0^x [1 + f'^2(x)]^{1/2}\, dx \leqslant |x|\cdot(1 - k^2)^{-1/2}$$

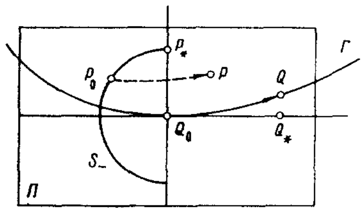
$$|f(x)| \leqslant x^2/\rho$$

hold for function $f(x)$.

**Proof.** We introduce a Cartesian coordinate system as follows: point $Q_0$ is the origin, the axis $Ox$ coincides with the tangent to $\Gamma$ at point $Q_0$, and the positive semiaxis $Oy$ is directed toward the center of curvature if the curvature of curve $\Gamma$ at point $Q_0$ is nonzero or is otherwise arbitrary (Figure).

By virtue of the regularity of curve $\Gamma$ it can be represented in every case by the graph of a function $y = f(x)$ on some interval $(-\varepsilon, \varepsilon)$. For values of $x$ taken from this interval, using Assumption A and integrating, we find

*Prikl.Matem.Mekhan.,46,No.4,pp.694-696,1982




$$-x/\rho \leqslant f'(x)[1 + f'^2(x)]^{-1/2} \leqslant x/\rho$$

Hence follows the first inequality in (2.1) for $|x| < \varepsilon$. But if the first inequality in (2.1) is fulfilled on some interval containing the point $x = 0$, then the function $y = f(x)$ has a bounded derivative and the curve $\Gamma$ cannot go outside the sector formed by the straight lines

$$y = \pm k(1 - k^2)^{-1/2}x$$

Therefore, the representation of curve $\Gamma$ as the graph of function $y = f(x)$ can be continued onto the segment $|x| \leqslant k\rho$ since $k \leqslant 1/\sqrt{5}$.

3. Escaper's strategy. We introduce a number $\delta > 0$ as follows: if $P_0Q_0 \leqslant k^2\rho$, then $\delta = P_0Q_0$, otherwise $\delta = P_0Q_0$. (*). We consider the circle $S: x^2 + y^2 = \delta^2$. If the inequality $P_0Q_0 > k^2\rho$ obtains for the initial positions $P_0$ and $Q_0$ of the pursuer and escaper, respectively, then we set the escaper's velocity equal to zero until $PQ = k^2\rho$ occurs. Here and subsequently $P$ and $Q$ are the players' positions at the current time $t$ and $PQ$ is the distance between points $P$ and $Q$. If, however, $P_0Q_0 \leqslant k^2\rho$, then by definition $P_0 \in S$. Thus, we can take it that $P_0 \in S$. To determine the escape direction on the circle $S$ we divide the normal to curve $\Gamma$ at point $Q_0$ into two semicircles $S_\pm: x = \pm(\delta^2 - y^2)^{1/2}$. For definiteness let $P_0 \in S_-$. Then the escaper is instructed to move away from point $Q_0$ towards the side of increase of $x$, i.e., along the part $y = f(x)$, $x \geqslant 0$, of curve $\Gamma$ (up to the point $k\rho, f(k\rho)$).

4. Estimate of the distance $PQ$. Lemma 2. Let $P_0 = (x_0, y_0)$. We set $P_* = (0, \delta)$ when $y_0 \geqslant 0$ and $P_* = (0, -\delta)$ when $y_0 < 0$. Then $P_0Q \geqslant P_*Q$ for any point $Q = (x, f(x)) \in \Gamma$, $0 \leqslant x \leqslant k\rho$.

Proof. In view of the total symmetry, henceforth we reckon that $y_0 \geqslant 0$. We have $|f(x)| \leqslant x$ (see the last inequality in (2.1)) and

$$|y_0 - \delta| = \delta - (\delta^2 - x_0^2)^{1/2} \leqslant -x_0, x_0 \leqslant 0$$

Therefore, $f(x)(y_0 - \delta) = |f(x)||y_0 - \delta| \leqslant -xx_0$, $xx_0 + f(x)y_0 \leqslant f(x)\delta$. The last inequality is equivalent to the one required.

Lemma 3. Let an instant $T$ be defined by the condition: $Q = (k\rho, f(k\rho))$ when $t = T$. Then

$$PQ \geqslant \delta^2 \text{ for } 0 \leqslant t \leqslant T,\ PQ \leqslant \delta \text{ for } t = T \tag{4.1}$$

Proof. With due regard to Lemma 2 we have

$$PQ \geqslant P_0Q - PP_0 \geqslant P_*Q - PP_0$$

But from Lemma 1 ensues

$$PP_0 \leqslant t = \frac{s(x)}{\sigma} \leqslant \frac{x}{\sigma}(1 - k^2)^{-1/2} \leqslant \frac{1 + k}{\sigma}x$$

Since $1 \leqslant (1 + k)^2(1 - k^2)$. In addition

$$P_*Q \geqslant P_*Q_* - QQ_*,\quad Q_* = (x, 0)$$

Consequently, to prove the first inequality in (4.1) it is enough to derive

$$(x^2 + \delta^2)^{1/2} - (1 + k)\sigma^{-1}x - f(x) \geqslant \delta^2 \tag{4.2}$$

Allowing for the conditions $|x| \leqslant k\rho \leqslant 1/\sqrt{5}$ and $\delta \leqslant k^2\rho \leqslant 1/5$, it can be shown that $(x^2 + \delta^2)^{1/2} \geqslant x + \delta^2$. Therefore, inequality (4.2) follows from the relation

$$x \geqslant (1 + k)\sigma^{-1}x + x^2\rho^{-1}$$

which, in its turn, is a consequence of the condition $k \leqslant (\sigma - 1)/(\sigma + 1)^{-1}$. Thus, the estimate $PQ \geqslant \delta^2$ for $t \subset [0, T]$ is fulfilled for

$$k \leqslant \min\left\{\frac{1}{\sqrt{5}}, \frac{\sigma - 1}{\sigma + 1}\right\}$$

Analogously to the above, the proof of the second inequality in (4.1) is reduced to the verification of the inequality

$$(x^2 + \delta^2)^{1/2} \geqslant T + \rho^{-1}s^2 + \delta \tag{4.3}$$

Taking into account that

$$x = k\rho,\ T = \sigma^{-1}s(k\rho) \leqslant k\rho\sigma^{-1}$$

(see the second inequality in (2.1)), we replace (4.3) by the stronger inequality

$$(k^2\rho^2 + \delta^2)^{1/2} \geqslant k\sigma^{-1}(1 - k^2)^{-1/2}\rho + k^2\rho + \delta$$

*) This appears in the original text - Ed.

We square this inequality and replace $\delta$ by the larger quantity $k^2\rho$. Then after cancellations we have

$$1 \geqslant \sigma^{-2}(1-k^2)^{-1} + 3k^2 + 4k\sigma^{-1}(1-k^2)^1$$

The inequality obtained is a corollary of

$$1 \geqslant \sigma^{-2} + 4k^2 + 4k\sigma^{-1}$$

which, in its own turn, follows from the condition $k \leqslant (\sigma-1)(2\sigma)^{-1}$. Thus, to complete the proof of Lemma 3 it suffices to take

$$k = \min\left\{\frac{1}{\sqrt{5}}, \frac{\sigma-1}{2\sigma}\right\} \tag{4.4}$$

5. Conclusion. If the escaper adopts the strategy constructed in Sect.3, then in the instant of time $t = T$ ensure the second estimate (4.2). This allows the further continuation of the escape process. Since each time the escaper goes on a path of length $s(k\rho)$ bounded from below by the constant $k\rho$ (see the second inequality in (2.1)), by virtue of Assumption B escape is possible for all $t \geqslant 0$. By the same token we have proved the following theorem (see the definition of $\delta$, and the remark after Assumption A).

Theorem. Let Assumptions A and B be fulfilled in the game being examined. Then escape is possible from any initial positions $P_0, Q_0, P_0 \neq Q_0$. The escape process can be carried out such that the following estimate for the distance $PQ$ between escaper and pursuer is observed:

$$PQ \geqslant (P_0Q_0)^2, \text{ if } P_0Q_0 < k^2 \min\{1, \rho\}$$
$$PQ \geqslant k^4 \min\{1, \rho^2\}, \text{ if } P_0Q_0 \geqslant k^2 \min\{1, \rho\}$$

where $k$ is determined by formula (4.4).

From the proof of the second inequality in (4.1) of Lemma 3 we can note that in case $P_0Q_0 < k^2 \min\{1, \rho\}$ we can ensure the estimate

$$PQ \geqslant k^4 \min\{1, \rho^2\}$$

beginning with the instant $T = s(k\rho)/\sigma$

If the curvature of curve $\Gamma$ is not bounded, then the escaper cannot always ensure that the distance $PQ$ can be estimate from below by a positive constant. For example, if as $\Gamma$ we take a hyperbolic spiral or the graph of the function $y = x\sin(1/x)$, $x > 0$, then by starting from many initial positions the pursuer can get arbitrarily close to the escaper. On the other hand, the theorem's proof remains in force for regular curves having selfintersections. An analogous theorem can be proved for space curves.

REFERENCES

1. CHERNOUS'KO F.L., A problem of evasion from many pursuers. PMM Vol.40, No.1, 1976.
2. FLYNN J.O., Lion and Man: the boundary constraint. SIAM J. Control, Vol.11, No.3, 1973.

Translated by N.H.C.